14-2 Translations of Trigonometric Graphs

Main Ideas

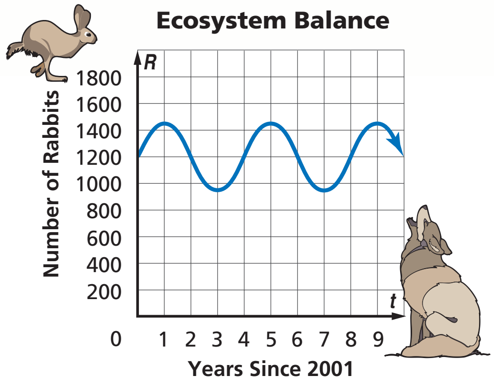
- Graph horizontal translations of trigonometric graphs and find phase shifts.
- Graph vertical translations of trigonometric graphs.

New Vocabulary

phase shift
vertical shift
midline

GET READY for the Lesson

In predator-prey ecosystems, the number of predators and the number of prey tend to vary in a periodic manner. In a certain region with coyotes as predators and rabbits as prey, the rabbit population R can be modeled by the equation $R = 1200 + 250 \sin \frac{1}{2}\pi t$, where t is the time in years since January 1, 2001.



Horizontal Translations Recall that a translation is a type of transformation in which the image is identical to the preimage in all aspects except its location on the coordinate plane. A horizontal translation shifts to the left or right, and not upward or downward.

GRAPHING CALCULATOR

Horizontal Translations

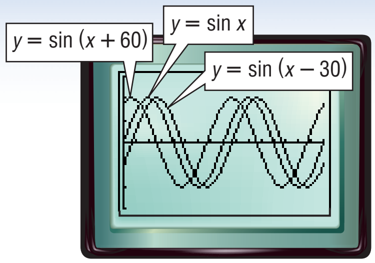
On a TI-83/84 Plus, set the MODE to degrees.

THINK AND DISCUSS

1. Graph $y = \sin x$ and $y = \sin (x - 30)$. How do the two graphs compare?

2. Graph $y = \sin (x + 60)$. How does this graph compare to the other two?

3. What conjecture can you make about the effect of h in the function $y = \sin (x - h)$?



[0, 720] scl: 45 by [−1.5, 1.5] scl: 0.5

4. Test your conjecture on the following pairs of graphs.

- $y = \cos x$ and $y = \cos (x + 30)$
- $y = \tan x$ and $y = \tan (x - 45)$
- $y = \sec x$ and $y = \sec (x + 75)$

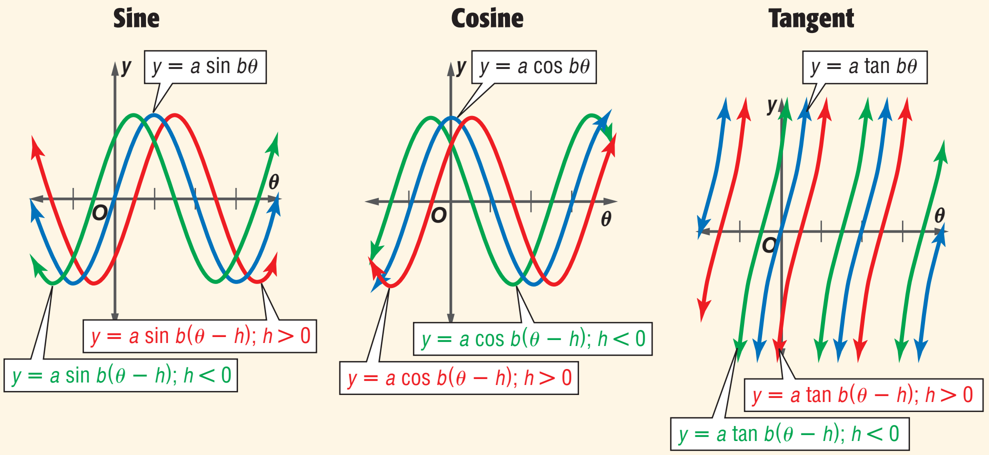
Notice that when a constant is added to an angle measure in a trigonometric function, the graph is shifted to the left or to the right. If (x, y) are coordinates of $y = \sin x$, then $(x \pm h, y)$ are coordinates of $y = \sin (x \mp h)$. A horizontal translation of a trigonometric function is called a **phase shift.**

KEY CONCEPT — Phase Shift

Words The phase shift of the functions $y = a \sin b(\theta - h)$, $y = a \cos b(\theta - h)$, and $y = a \tan b(\theta - h)$ is h, where $b > 0$.

If $h > 0$, the shift is to the right.
If $h < 0$, the shift is to the left.

Models:



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The secant, cosecant, and cotangent can be graphed using the same rules.

Study Tip

Verifying a Graph

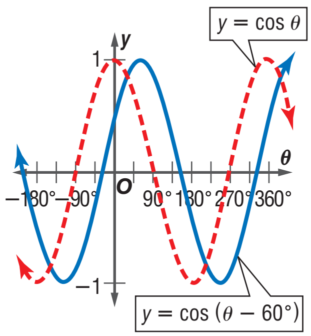
After drawing the graph of a trigonometric function, select values of θ and evaluate them in the equation to verify your graph.

EXAMPLE Graph Horizontal Translations

1 State the amplitude, period, and phase shift for $y = \cos(\theta - 60°)$. Then graph the function.

Since $a = 1$ and $b = 1$, the amplitude and period of the function are the same as $y = \cos\theta$. However, $h = 60°$, so the phase shift is 60°. Because $h > 0$, the parent graph is shifted to the right.

To graph $y = \cos(\theta - 60°)$, consider the graph of $y = \cos\theta$. Graph this function and then shift the graph 60° to the right. The graph $y = \cos(\theta - 60°)$ is the graph of $y = \cos\theta$ shifted to the right.



CHECK Your Progress

1. State the amplitude, period, and phase shift for $y = 2\sin\left(\theta + \frac{\pi}{4}\right)$. Then graph the function.

Study Tip

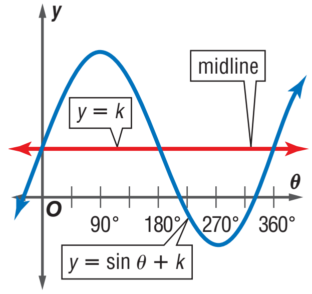
Notation

Pay close attention to trigonometric functions for the placement of parentheses. Note that $\sin(\theta + x) \neq \sin\theta + x$. The first expression represents a phase shift while the second expression represents a vertical shift.

Vertical Translations In Chapter 5, you learned that the graph of $y = x^2 + 4$ is a vertical translation of the parent graph of $y = x^2$. Similarly, graphs of trigonometric functions can be translated vertically through a **vertical shift**.

When a constant is added to a trigonometric function, the graph is shifted upward or downward. If (x, y) are coordinates of $y = \sin x$, then $(x, y \pm k)$ are coordinates of $y = \sin x \pm k$.

A new horizontal axis called the **midline** becomes the reference line about which the graph oscillates. For the graph of $y = \sin\theta + k$, the midline is the graph of $y = k$.



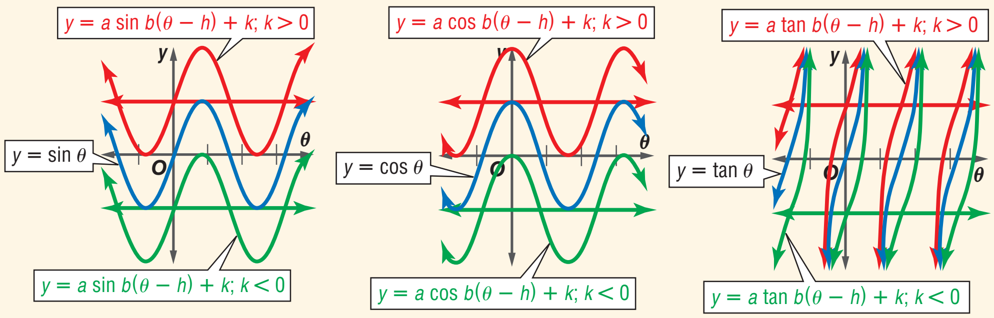
KEY CONCEPT — Vertical Shift

Words The vertical shift of the functions $y = a\sin b(\theta - h) + k$, $y = a\cos b(\theta - h) + k$, and $y = a\tan b(\theta - h) + k$ is k.

If $k > 0$, the shift is up. If $k < 0$, the shift is down. The midline is $y = k$.

Models:

Sine **Cosine** **Tangent**



The secant, cosecant, and cotangent can be graphed using the same rules.

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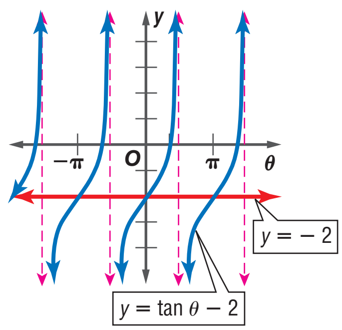
EXAMPLE Graph Vertical Translations

2 **State the vertical shift, equation of the midline, amplitude, and period for $y = \tan\theta - 2$. Then graph the function.**

Since $\tan\theta - 2 = \tan\theta + (-2)$, $k = -2$, and the vertical shift is -2. Draw the midline, $y = -2$. The tangent function has no amplitude and the period is the same as that of $\tan\theta$.

Draw the graph of the function relative to the midline.



Study Tip

Graphing

It may be helpful to first graph the parent graph $y = \sin\theta$ in one color. Then apply the vertical shift and graph the function in another color. Then apply the change in amplitude and graph the function in the final color.

CHECK Your Progress

2. State the vertical shift, equation of the midline, amplitude, and period for $y = \frac{1}{2}\sin\theta + 1$. Then graph the function.

In general, use the following steps to graph any trigonometric function.

CONCEPT SUMMARY *Graphing Trigonometric Functions*

Step 1 Determine the vertical shift, and graph the midline.

Step 2 Determine the amplitude, if it exists. Use dashed lines to indicate the maximum and minimum values of the function.

Step 3 Determine the period of the function and graph the appropriate function.

Step 4 Determine the phase shift and translate the graph accordingly.

EXAMPLE Graph Transformations

3 **State the vertical shift, amplitude, period, and phase shift of $y = 4\cos\left[\frac{1}{2}\left(\theta - \frac{\pi}{3}\right)\right] - 6$. Then graph the function.**

The function is written in the form $y = a\cos[b(\theta - h)] + k$. Identify the values of k, a, b, and h.

$k = -6$, so the vertical shift is -6.

$a = 4$, so the amplitude is $|4|$ or 4.

$b = \frac{1}{2}$, so the period is $\frac{2\pi}{\left|\frac{1}{2}\right|}$ or 4π.

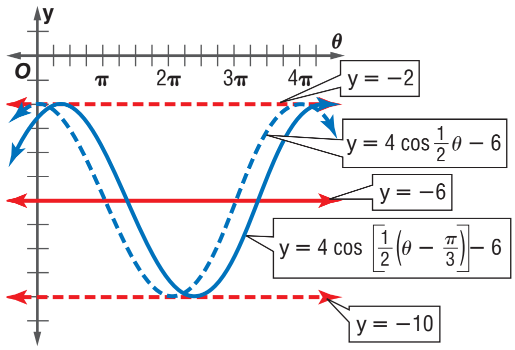
$h = \frac{\pi}{3}$, so the phase shift is $\frac{\pi}{3}$ to the right.

Step 1 The vertical shift is -6. Graph the midline $y = -6$.

Step 2 The amplitude is 4. Draw dashed lines 4 units above and below the midline at $y = -2$ and $y = -10$.

Step 3 The period is 4π, so the graph will be stretched. Graph $y = 4\cos\frac{1}{2}\theta - 6$ using the midline as a reference.

Step 4 Shift the graph $\frac{\pi}{3}$ to the right.



CHECK Your Progress **Graph each equation.**

3. State the vertical shift, amplitude, period, and phase shift of $y = 3\sin\left[\frac{1}{3}\left(\theta - \frac{\pi}{2}\right)\right] + 2$. Then graph the function.

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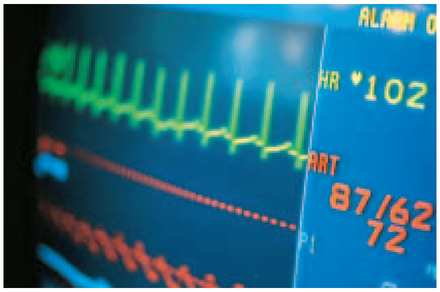
Real-World EXAMPLE

4 **HEALTH Suppose a person's resting blood pressure is 120 over 80. This means that the blood pressure oscillates between a maximum of 120 and a minimum of 80. If this person's resting heart rate is 60 beats per minute, write a sine function that represents the blood pressure at time *t* seconds. Then graph the function.**



Real-World Link

Blood pressure can change from minute to minute and can be affected by the slightest of movements, such as tapping your fingers or crossing your arms.

Source: American Heart Association

Explore You know that the function is periodic and can be modeled using sine.

Plan Let P represent blood pressure and let t represent time in seconds. Use the equation $P = a \sin [b(t - h)] + k$.

Solve

- Write the equation for the midline. Since the maximum is 120 and the minimum is 80, the midline lies halfway between these values.

  $P = \frac{120 + 80}{2}$ or 100

- Determine the amplitude by finding the difference between the midline value and the maximum and minimum values.

  $a = |120 - 100|$ $\quad = |20|$ or 20 $\qquad$ $a = |80 - 100|$ $\quad = |-20|$ or 20

  Thus, $a = 20$.

- Determine the period of the function and solve for b. Recall that the period of a function can be found using the expression $\frac{2\pi}{|b|}$.

  Since the heart rate is 60 beats per minute, there is one heartbeat, or cycle, per second. So, the period is 1 second.

  $1 = \frac{2\pi}{|b|}$ Write an equation.

  $|b| = 2\pi$ Multiply each side by $|b|$.

  $b = \pm 2\pi$ Solve.

  For this example, let $b = 2\pi$. The use of the positive or negative value depends upon whether you begin a cycle with a maximum value (positive) or a minimum value (negative).

- There is no phase shift, so $h = 0$. So, the equation is $P = 20 \sin 2\pi t + 100$.

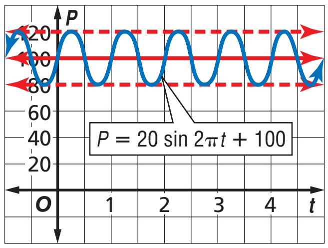
- Graph the function.

  Step 1 Draw the midline $P = 100$.

  Step 2 Draw maximum and minimum reference lines.

  Step 3 Use the period to draw the graph of the function.

  Step 4 There is no phase shift.



Check Notice that each cycle begins at the midline, rises to 120, drops to 80, and then returns to the midline. This represents the blood pressure of 120 over 80 for one heartbeat. Since each cycle lasts 1 second, there will be 60 cycles, or heartbeats, in 1 minute. Therefore, the graph accurately represents the information.

CHECK Your Progress

4. Suppose that while doing some moderate physical activity, the person's blood pressure is 130 over 90 and that the person has a heart rate of 90 beats per minute. Write a sine function that represents the person's blood pressure at time *t* seconds. Then graph the function.

CHECK Your Understanding

Example 1 (p. 830)

State the amplitude, period, and phase shift for each function. Then graph the function.

1. $y = \sin\left(\theta - \frac{\pi}{2}\right)$

2. $y = \tan(\theta + 60°)$

3. $y = \cos(\theta - 45°)$

4. $y = \sec\left(\theta + \frac{\pi}{3}\right)$

Example 2 (pp. 831–832)

State the vertical shift, equation of the midline, amplitude, and period for each function. Then graph the function.

5. $y = \cos\theta + \frac{1}{4}$

6. $y = \sec\theta - 5$

7. $y = \tan\theta + 4$

8. $y = \sin\theta + 0.25$

Example 3 (p. 832)

State the vertical shift, amplitude, period, and phase shift for each function. Then graph the function.

9. $y = 3\sin[2(\theta - 30°)] + 10$

10. $y = 2\cot(3\theta + 135°) - 6$

11. $y = \frac{1}{2}\sec\left[4\left(\theta - \frac{\pi}{4}\right)\right] + 1$

12. $y = \frac{2}{3}\cos\left[\frac{1}{2}\left(\theta + \frac{\pi}{6}\right)\right] - 2$

Example 4 (p. 833)

PHYSICS For Exercises 13—15, use the following information.

A weight is attached to a spring and suspended from the ceiling. At equilibrium, the weight is located 4 feet above the floor. The weight is pulled down 1 foot and released.

13. Determine the vertical shift, amplitu de, and period of a function that represents the height of the weight above the floor if the weight returns to its lowest position every 4 seconds.

14. Write the equation for the height *h* of the weight above the floor as a function of time *t* seconds.

15. Draw a graph of the function you wrote in Exercise 14.

Exercises

HOMEWORK HELP

For Exercises	See Examples
16–21	1
22–27	2
28–35	3
36–38	4

State the amplitude, period, and phase shift for each function. Then graph the function.

16. $y = \cos(\theta + 90°)$

17. $y = \cot(\theta - 30°)$

18. $y = \sin\left(\theta - \frac{\pi}{4}\right)$

19. $y = \cos\left(\theta + \frac{\pi}{3}\right)$

20. $y = \frac{1}{4}\tan(\theta + 22.5°)$

21. $y = 3\sin(\theta - 75°)$

State the vertical shift, equation of the midline, amplitude, and period for each function. Then graph the function.

22. $y = \sin\theta - 1$

23. $y = \sec\theta + 2$

24. $y = \cos\theta - 5$

25. $y = \csc\theta - \frac{3}{4}$

26. $y = \frac{1}{2}\sin\theta + \frac{1}{2}$

27. $y = 6\cos\theta + 1.5$

State the vertical shift, amplitude, period, and phase shift for each function. Then graph the function.

28. $y = 2 \sin [3(\theta - 45°)] + 1$

29. $y = 4 \cos [2(\theta + 30°)] - 5$

30. $y = 3 \csc \left[\frac{1}{2}(\theta + 60°)\right] - 3.5$

31. $y = 6 \cot \left[\frac{2}{3}(\theta - 90°)\right] + 0.75$

32. $y = \frac{1}{4} \cos (2\theta - 150°) + 1$

33. $y = \frac{2}{5} \tan (6\theta + 135°) - 4$

34. $y = 3 + 2 \sin \left[\left(2\theta + \frac{\pi}{4}\right)\right]$

35. $y = 4 + 5 \sec \left[\frac{1}{3}\left(\theta + \frac{2\pi}{3}\right)\right]$

Real-World Link

The average weight of a male Cactus Ferruginous Pygmy-Owl is 2.2 ounces.

Source: www.kidsplanet.org

ZOOLOGY **For Exercises 36–38, use the following information.**

The population of predators and prey in a closed ecological system tends to vary periodically over time. In a certain system, the population of owls O can be represented by $O = 150 + 30 \sin \left(\frac{\pi}{10}t\right)$ where t is the time in years since January 1, 2001. In that same system, the population of mice M can be represented by $M = 600 + 300 \sin \left(\frac{\pi}{10}t + \frac{\pi}{20}\right)$.

36. Find the maximum number of owls. After how many years does this occur?

37. What is the minimum number of mice? How long does it take for the population of mice to reach this level?

38. Why would the maximum owl population follow behind the population of mice?

39. Graph $y = 3 - \frac{1}{2} \cos \theta$ and $y = 3 + \frac{1}{2} \cos (\theta + \pi)$. How do the graphs compare?

40. Compare the graphs of $y = -\sin \left[\frac{1}{4}\left(\theta - \frac{\pi}{2}\right)\right]$ and $y = \cos \left[\frac{1}{4}\left(\theta + \frac{3\pi}{2}\right)\right]$.

41. Graph $y = 5 + \tan \left(\theta + \frac{\pi}{4}\right)$. Describe the transformation to the parent graph $y = \tan \theta$.

42. Draw a graph of the function $y = \frac{2}{3}\cos (\theta - 50°) + 2$. How does this graph compare to the graph of $y = \cos \theta$?

EXTRA PRACTICE

See pages 922, 939.

Math Online

Self-Check Quiz at algebra2.com

43. MUSIC When represented on oscilloscope, the note A above middle C has a period of $\frac{1}{440}$. Which of the following can be an equation for an oscilloscope graph of this note? The amplitude of the graph is K.

a. $y = K \sin 220\pi t$ **b.** $y = K \sin 440\pi t$ **c.** $y = K \sin 880\pi t$

44. TIDES The height of the water in a harbor rose to a maximum height of 15 feet at 6:00 P.M. and then dropped to a minimum level of 3 feet by 3:00 A.M. Assume that the water level can be modeled by the sine function. Write an equation that represents the height h of the water t hours after noon on the first day.

H.O.T. Problems

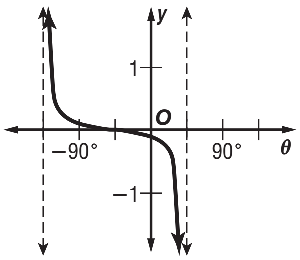
45. OPEN ENDED Write the equation of a trigonometric function with a phase shift of $-45°$. Then graph the function, and its parent graph.

46. CHALLENGE The graph of $y = \cot \theta$ is a transformation of the graph of $y = \tan \theta$. Determine a, b, and h so that $\cot \theta = a \tan [b(\theta - h)]$ for all values of θ for which each function is defined.

47. ***Writing in Math*** Use the information on page 829 to explain how translations of trigonometric graphs can be used to show animal populations. Include a description of what each number in the equation $R = 1200 + 250 \sin \frac{1}{2}\pi t$ represents.

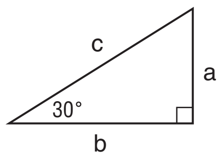
STANDARDIZED TEST PRACTICE

48. ACT/SAT Which equation is represented by the graph?



A $y = \cot(\theta + 45°)$
B $y = \cot(\theta - 45°)$
C $y = \tan(\theta + 45°)$
D $y = \tan(\theta - 45°)$

49. REVIEW Refer to the figure below. If $c = 14$, find the value of b.



F $\frac{\sqrt{3}}{2}$
G $14\sqrt{3}$
H 7
J $7\sqrt{3}$

Spiral Review

Find the amplitude, if it exists, and period of each function. Then graph each function. (Lesson 14-1)

50. $y = 3 \csc \theta$ **51.** $y = \sin \frac{\theta}{2}$ **52.** $y = 3 \tan \frac{2}{3}\theta$

Find each value. (Lesson 13-7)

53. $\sin\left(\text{Cos}^{-1} \frac{2}{3}\right)$ **54.** $\cos\left(\text{Cos}^{-1} \frac{4}{7}\right)$ **55.** $\text{Sin}^{-1}\left(\sin \frac{5}{6}\right)$ **56.** $\cos\left(\text{Tan}^{-1} \frac{3}{4}\right)$

57. GEOMETRY Find the total number of diagonals that can be drawn in a decagon. (Lesson 12-2)

Solve each equation. Round to the nearest hundredth. (Lesson 9-4)

58. $4^x = 24$ **59.** $4.3^{3x+1} = 78.5$ **60.** $7^{x-2} = 53^{-x}$

Simplify each expression. (Lesson 8-4)

61. $\frac{3}{a-2} + \frac{2}{a-3}$ **62.** $\frac{w+12}{4w-16} - \frac{w+4}{2w-8}$ **63.** $\frac{3y+1}{2y-10} + \frac{1}{y^2-2y-15}$

GET READY for the Next Lesson

PREREQUISITE SKILL Find the value of each function. (Lessons 13-3)

64. $\cos 150°$ **65.** $\tan 135°$ **66.** $\sin \frac{3\pi}{2}$ **67.** $\cos\left(-\frac{\pi}{3}\right)$

68. $\sin(-\pi)$ **69.** $\tan\left(-\frac{5\pi}{6}\right)$ **70.** $\cos 225°$ **71.** $\tan 405°$